Quantum-like approaches to the beam halo problem *

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Abstract

An interpretation of the the “halo problem” in accelerators based on quantum-like diffraction is given. Comparison between this approach and the others based on classical mechanics equations is discussed
Keywords: Beam Physics, Quantum-like, Beam halo, Beam Losses, Stochasticity.

I. INTRODUCTION

Recently the description of the dynamical evolution of high density beams by using the collective models, has become more and more popular. A way of developing this point of view is the quantum-like approach [1] where one considers a time-dependent Schrödinger equation, in both the usual linear and the less usual nonlinear forms, as a fluid equation for the whole beam. In this case the squared modulus of the wave function (named beam wave function) gives the distribution function of the particles in space at a certain time [2]. The Schrödinger equation may be taken in one or more spacial dimensions according to the particular physical problem; furthermore the fluid becomes a Madelung fluid if one choses the equation in its usual linear version.

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Although the validity of the model relies only on experiments and in particular on new predictions which must be verified experimentally, we like to invoke here a theoretical argument that could justify the Schrödinger quantum-like approach we are going to apply. Let us think of particles in motion within a bunch in such a way that the single particle moves under an average force field due to the presence of others and collides with the neighbouring ones in a complicated manner. It is obviously impossible to follow and predict all the forces deterministically. We then face a situation where the classical motion determined by the force-field is perturbed continuously by a random term, and we have a connection with a stochastic process. If one simply assumes that the process is Markovian and Brownian, one obtains following Nelson [3], a modification of the classical equations of motion that can be synthesized by a linear Schrödinger equation which depends on a physical parameter having the dimension of action [4]. Usual wave quantum mechanics follows if this parameter is chosen as the Planck's constant $\hbar$, whereas the quantum-like theory of beams in the linearized version is obtained if one choses the normalized emittance ϵ [1]. In both cases, quantum particle and beam respectively, the evolution of the system is expressed in terms of a continuous field ψ defining the so called Madelung fluid. We may notice that the normalized emittance ϵ having the dimension of action is the natural choice for the parameter in the quantum-like theory that corresponds to Planck's constant $\hbar$ in the quantum theory, because it reproduces the corresponding area in the phase-space of the particle.

We here point out that, after linearizing the Schrödinger-like equation, for beams in an accelerator one can use the whole apparatus of quantum mechanics, with a new interpretation of the basic parameters (for instance the Planck's constant $\hbar \longrightarrow \epsilon$ where ϵ is the normalized beam emittance) and introduce the propagator $K\left(x_f, t_f | x_i, t_i\right)$ of the Feynman theory for both longitudinal and transversal motion. A procedure of this sort seems particularly effective for a global description of several phenomena such as intrabeam scattering, space-charge, particle focusing, that cannot be treated easily in detail by "classical mechanics" and are considered to be the main cause for the creation of the *Halo* around the beam line with consequent losses of particles.

Let us indeed consider the Schrödinger like equation for the beam wave function

$$\mathrm{i}\epsilon\partial_t\psi = -\frac{\epsilon^2}{2m}\partial_x^2\psi + U\left(x,t\right)\psi \tag{1}$$

in the linearized case $U\left(x,t\right)$ does not depend on the density $\left|\psi\right|^2$. ϵ here is the normalized transversal beam emittance defined as follows:

$$\epsilon = m_0 c\gamma\beta\tilde{\epsilon}\,, \tag{2}$$

$\tilde{\epsilon}$ being the emittance usually considered, where as we may introduce the analog of the de Broglie wavelength as $\lambda = \epsilon/p$. We now focus our attention on the one dimensional transversal motion along the x-axis of the beam particles belonging to a single bunch and assume a Gaussian transversal profile for a particles injected in to a circular machine. We describe all the interactions mentioned above, that cannot be treated in detail, as diffraction effects by a phenomenological boundary defined by a slit, in each segment of the particle trajectory. This condition should be applied to both beam wave function and its corresponding beam propagator K. The result of such a procedure is a multiple integral that determines the

actual propagator between the initial and final states in terms of the space-time intervals due to the intermediate segments.

$$K\left(x+x_0,T+\tau|x',0\right) = \int_{-b}^{+b} K\left(x+x_0,\tau|x_0+y_n,T+(n-1)\tau'\right) \times K\left(x+y_n,T+(n-1)\tau'|x_0+y_{n-1},T+(n-2)\tau'\right) \times \cdots K\left(x+y_1,T|x',0\right)dy_1dy_2\cdots dy_n \tag{3}$$

where $\tau = n\tau'$ is the total time of revolutions T is the time necessary to insert the bunch (practically the time between two successive bunches) and $(-b,+b)$ the space interval defining the boundary conditions. Obviously b and T are phenomenological parameters which vary from a machine to another and must also be correlated with the geometry of the vacuum tube where the particles circulate.

At this point we may consider two possible approximations for $K\left(n|n-1\right) \equiv K\left(x_0+y_n,T+(n-1)\tau'|x_0+y_{n-1}+(n-2)\tau'\right)$:

1. We substitute it with the free particle K_0 assuming that in the τ' interval ($\tau' \ll \tau$) the motion is practically a free particle motion between the boundaries $(-b,+b)$.

2. We substitute it with the harmonic oscillator $K_\omega\left(n|n-1\right)$ considering the harmonic motion of the betatronic oscillations with frequency $\omega/2\pi$

II. FREE PARTICLE CASE

We may notice that the convolution property (3) of the Feynman propagator allows us to substitute the multiple integral (that becomes a functional integral for $n \longrightarrow \infty$ and $\tau' \longrightarrow 0$) with the single integral

$$K\left(x+x_0,T+\tau|x',0\right) = \int_{-b}^{+b} dyK\left(x+x_0,T+\tau|x_0+y,T\right)K\left(x_0+y,T|x',0\right)dy \tag{4}$$

In this note we mainly discuss the case 1. and obtain from equation (4) after introducing the Gaussian slit $\exp\left[-\frac{y^2}{2b^2}\right]$ instead of the segment $(-b,+b)$ we obtain from

$$\begin{aligned}
&K\left(x+x_0,T+\tau|x',0\right) \\
&= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2}\right]\left\{\frac{2\pi \mathrm{i}\hbar\tau}{m}\frac{2\pi \mathrm{i}\hbar T}{m}\right\}^{-\frac{1}{2}} \exp\left[\frac{im}{2\hbar\tau}(x-y)^2\right]\exp\left[\frac{im}{2\hbar T}(x_0+y-x')^2\right] \\
&= \sqrt{\frac{m}{2\pi \mathrm{i}\hbar}}\left(T+\tau+T\tau\frac{\mathrm{i}\hbar}{mb^2}\right)^{-\frac{1}{2}} \exp\left[\frac{im}{2\hbar}\left(v_0^2T+\frac{x^2}{\tau}\right)+\frac{\left(m^2/2\hbar^2\tau^2\right)(x-v_0\tau)^2}{\frac{im}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right]
\end{aligned} \tag{5}$$

where $v_0 = \frac{x_0-x'}{T}$ and $x_0$is the initial central point of the beam at injection and can be chosen as the origin ($x_0=0$) of the transverse motion of the reference trajectory in the test particle reference frame. **Where as $\hbar$ must be interpreted as the normalized beam emittance in the quantum-like approach**.

With an initial Gaussian profile (at $t=0$), the beam wave function (normalized to 1) is

$$f(x) = \left\{ \frac{\alpha}{\pi} \right\}^{\frac{1}{4}} \exp\left[-\frac{\alpha}{2} x'^2 \right] \tag{6}$$

r.m.s of the transverse beam and the final beam wave function is:

$$\phi(x) = \int_{-\infty}^{+\infty} dx' \left(\frac{\alpha}{\pi} \right)^{\frac{1}{4}} e^{\left[-\frac{\alpha}{2} x'^2 \right]} K\left(x, T+\tau; x', 0 \right) = B \exp\left[C x^2 \right] \tag{7}$$

with

$$B = \sqrt{\frac{m}{2\pi \mathrm{i}\hbar}} \left\{ T + \tau + T\tau \frac{\mathrm{i}\hbar}{mb^2} \right\}^{-\frac{1}{2}} \left\{ \frac{\alpha}{\pi} \right\}^{\frac{1}{4}} \sqrt{\frac{\pi}{\left(\frac{\alpha}{2} - \frac{\mathrm{i}m}{2\hbar T} - \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T} + \frac{1}{\tau} \right) - \frac{1}{b^2}} \right)}}$$

$$C = \frac{\mathrm{i}m}{2\hbar\tau} + \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T} + \frac{1}{\tau} \right) - \frac{1}{b^2}} + \frac{\frac{\tau^2}{T^2} \left\{ \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T} + \frac{1}{\tau} \right) - \frac{1}{b^2}} \right\}^2}{\left(\frac{\alpha}{2} - \frac{\mathrm{i}m}{2\hbar T} - \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T} + \frac{1}{\tau} \right) - \frac{1}{b^2}} \right)} \tag{8}$$

The final local distribution of the beam that undergoes the diffraction is therefore

$$\rho(x) = |\phi(x)|^2 = BB^* \exp\left[-\tilde{\alpha} x^2 \right] \tag{9}$$

where $\tilde{\alpha} = -(C + C^*)$ and the total probability per particle is given by

$$P = \int_{-\infty}^{+\infty} dx \rho(x) = BB^* \sqrt{\frac{\pi}{\tilde{\alpha}}} \approx \frac{1}{\sqrt{\alpha}} \frac{mb}{\hbar T} \tag{10}$$

One may notice that the probability P has the same order of magnitude of the one computed in [5] if $\frac{1}{\sqrt{\alpha}}$ is of the order of b.

III. OSCILLATOR CASE

Similarly we may consider the harmonic oscillator case (betatronic oscillations) to compute the diffraction probability of the single particle from the beam wave function and evaluate the probability of beam losses per particle. The propagator $K_\omega\left(x, T+\tau | y, T \right)$ in the later case is:

$$\begin{aligned}
& K\left(x, T+\tau | x', 0 \right) \\
&= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2} \right] K_\omega\left(x, T+\tau | y, T \right) K_\omega\left(y, T | x', 0 \right) \\
&= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2} \right] \left\{ \frac{m\omega}{2\pi \mathrm{i}\hbar \sin(\omega\tau)} \right\}^{\frac{1}{2}} \exp\left[\frac{\mathrm{i}m\omega}{2\hbar \sin(\omega\tau)} \left\{ \left(x^2 + y^2 \right) \cos\omega\tau - 2xy \right\} \right] \\
& \qquad \times \left\{ \frac{m\omega}{2\pi \mathrm{i}\hbar \sin(\omega T)} \right\}^{\frac{1}{2}} \exp\left[\frac{\mathrm{i}m\omega}{2\hbar \sin(\omega T)} \left\{ \left(y^2 + x'^2 \right) \cos\omega T - 2x'y \right\} \right] \\
&= \left\{ \frac{1}{2\pi} \tilde{C} \right\}^{\frac{1}{2}} \exp\left[\tilde{A} x^2 + \tilde{B} x'^2 + \tilde{C} x x' \right]
\end{aligned} \tag{11}$$

where

$$\tilde{A} = \mathrm{i}\frac{m\omega}{2\hbar}\frac{\cos(\omega\tau)}{\sin(\omega\tau)} - \left(\frac{m\omega}{2\hbar}\right)^2 \frac{1}{\sin^2(\omega\tau)}\frac{1}{D}, \qquad \tilde{B} = \mathrm{i}\frac{m\omega}{2\hbar}\frac{\cos(\omega T)}{\sin(\omega T)} - \left(\frac{m\omega}{2\hbar}\right)^2 \frac{1}{\sin^2(\omega T)}\frac{1}{D}$$

$$\tilde{C} = -\left(\frac{m\omega}{2\hbar}\right)^2 \frac{2}{\sin(\omega\tau)\sin(\omega T)}\frac{1}{D}, \qquad D = \frac{1}{2b^2} - \mathrm{i}\frac{m\omega}{2\hbar}\left(\frac{\cos(\omega\tau)}{\sin(\omega\tau)} + \frac{\cos(\omega T)}{\sin(\omega T)}\right) \quad (12)$$

$$\phi_\omega(x) = \int_{-\infty}^{+\infty} dx' \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{\alpha}{2}x'^2\right] K_\omega(x, T+\tau; x', 0) = N \exp\left[Mx^2\right] \quad (13)$$

where

$$N = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \left\{\frac{\tilde{C}}{\left(\alpha - 2\tilde{B}\right)}\right\}^{\frac{1}{2}}, \qquad M = \tilde{A} + \frac{\tilde{C}^2}{2\left(\alpha - 2\tilde{B}\right)} \quad (14)$$

$$\rho_\omega(x) = |\phi_\omega(x)|^2 = N^* N \exp\left[-\left(M^* + M\right)x^2\right] \quad (15)$$

$$P_\omega = \int_{-\infty}^{+\infty} dx \rho(x) = N^* N \sqrt{\frac{\pi}{(M^* + M)}} \approx \frac{1}{\sqrt{\alpha}}\frac{mb}{\hbar}\frac{\omega}{\sin(\omega T)} \quad (16)$$

From the approximate formulae 10 and 16 we notice that the parameter τ does not play a significant role in the calculation of the probabilities. We gave it a value $\tau = 1$ sec., considering about 10^6 revolutions in LHC and $HIDIF$ storage rings.

IV. PRELIMINARY ESTIMATES

TABLE-I: Free Particle Case

Parameters	**LHC**	**HIDIF**
Normalized Transverse Emittance	3.75 mm mrad	13.5 mm mrad
Total Energy, E	450 GeV	5 Gev
T	25 nano sec.	100 nano sec.
b	1.2 mm	1.0 mm
P	3.39×10^{-5}	2.37×10^{-3}

TABLE-II: Oscillator Case

Parameters	**LHC**	**HIDIF**
Normalized Transverse Emittance	3.75 mm mrad	13.5 mm mrad
Total Energy, E	450 GeV	5 Gev
T	25 nano sec.	100 nano sec.
b	1.2 mm	1.0 mm
ω	4.47×10^6 Hz	1.12×10^7 Hz
P_ω	3.44×10^{-5}	2.96×10^{-3}

V. CONCLUSION

The parameters entering into the probability formulae are very few and must be looked at as purely phenomenological. To be more specific, b, τ, and T (b in particular) hide several fundamental processes that may be present in the beam bunches and that play a deterministic role in the creation of the **halo** such as intrabeam scattering beamstrahlung, space-charge effects and imperfections in the magnets of the lattice that cause nonlinear perturbative effects.

The fact that such a small amount parameters take into account many physical processes is a nice feature of the quantum-like diffraction approach. However a way of connecting this method with the physical processes mentioned above as well as with the nonlinear dynamical classical theory is mandatory at this point.

Another interesting feature of the parameters used is that their numerical values are very reasonable because they are within the ranges. One expects: T may be considered as the average time interval between the two successive injection, τ the time interval between two successive diffractions ($\tau = n\tau'$ is the total time of revolutions) and $2b$ the phenomenological diffraction slit width. We recall that in the usual optics diffraction through a slit is also a macroscopic means of dealing with many complicated physical effects such as scattering of light, electrons etc., at the atomic level.

The two relevant concluding remarks are the following:

1. The probability calculated from the free and the harmonic oscillator propagators (both in the transversal motion of the particles) appear very close for the two different circular systems such as LHC and $HIDIF$ rings.

2. The $HIDIF$ scenario, as expected has a total loss of beam power which is at least 10^3 times higher than LHC.

These preliminary numerical results are encouraging because they predict halo losses which seem under control. Indeed the $HIDIF$ scenario gives a total loss of beam power per meter which is about a thousand higher than the LHC; however in both cases the estimated losses appear much smaller than the permissible 1 Watt/m.

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